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# More on the Burkhardt-Cottingham Sum Rule in QCD

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## Abstract

The QCD higher order effects to the polarized structure function  $g_2(x, Q^2)$  are reanalyzed for massive quarks in the context of the operator product expansion. We confirm that the lowest moment of  $g_2(x, Q^2)$  which corresponds to the Burkhardt-Cottingham sum rule does *not* suffer from radiative corrections in perturbative QCD.

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In the polarized leptonproduction, we have two structure functions  $g_1$  and  $g_2$ . The QCD effects on  $g_1$  have been extensively studied mainly in connection with the problem of so-called “spin crisis ”. On the other hand, a new measurement of  $g_2$  is expected to be performed at CERN, SLAC and DESY in the near future. Such experiment is very important since it has been known that the twist-3 operators as well as the twist-2 operators contribute to  $g_2$  [1] and furthermore we can check the well-known sum rule

$$\int_0^1 dx \, g_2(x, Q^2) = 0 \quad (1)$$

called the Burkhardt-Cottingham (BC) sum rule.

The fact that the twist-3 operators also contribute to the moment of  $g_2$  in the leading order of  $1/Q^2$  produces new features which do not appear in the analyses of other structure functions. At the higher-twist level, in general, the appearance of the composite operators in the operator product expansion (OPE) which are proportional to the *equation of motion* makes the operator mixing problems complicated [2]. This problem has been discussed by many authors [3, 4, 5] after the old papers [6, 7, 8] on the polarized process. However, as far as the first moment ( $n = 1$ ) of  $g_2$  is concerned, the above complexity is irrelevant since there is no operators corresponding to  $n = 1$ . Therefore it is naively expected that the BC sum rule is exact at least perturbatively.

Recently two groups [9, 10] have discussed the QCD effects at order  $\alpha_s$  to BC sum rule for massive quarks and reached different conclusions on the validity of BC sum rule Eq.(1). This controversial situation must be resolved especially for the experimentalists who plan the measurement of  $g_2$ . The purpose of this note is to present an independent calculation of the QCD corrections to BC sum rule in the framework of OPE and try to settle the above issue.

Spin-dependent structure functions are defined by the antisymmetric (A) part of the Fourier transform of the commutator of two electromagnetic currents sandwiched

between polarized nucleon states.

$$W_{\mu\nu} = \frac{1}{2\pi} \int d^4x e^{iq \cdot x} \langle p, s | [J_\mu(x), J_\nu(0)] | p, s \rangle \equiv W_{\mu\nu}^S + iW_{\mu\nu}^A,$$

where  $p$  ( $s$ ) is the nucleon's momentum (covariant spin) and  $q$  is the virtual photon momentum. We also introduce the current correlation function

$$T_{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p, s | T(J_\mu(x) J_\nu(0)) | p, s \rangle \equiv T_{\mu\nu}^S + iT_{\mu\nu}^A, \quad (2)$$

such that

$$W_{\mu\nu} = \frac{1}{\pi} \text{Im} T_{\mu\nu}.$$

The antisymmetric part  $W_{\mu\nu}^A$  is expressed in terms of two structure functions  $g_1$  and  $g_2$ .

$$W_{\mu\nu}^A = \varepsilon_{\mu\nu\lambda\sigma} q^\lambda \left\{ s^\sigma \frac{1}{p \cdot q} g_1(x, Q^2) + (p \cdot q s^\sigma - q \cdot s p^\sigma) \frac{1}{(p \cdot q)^2} g_2(x, Q^2) \right\},$$

where  $x$  is the Bjorken variable  $x = Q^2/2p \cdot q$  and  $q^2 = -Q^2$ . We will make the same tensor decomposition for also the current correlation function.

$$T_{\mu\nu}^A = \varepsilon_{\mu\nu\lambda\sigma} q^\lambda \left\{ s^\sigma \frac{1}{p \cdot q} t_1(\omega, Q^2) + (p \cdot q s^\sigma - q \cdot s p^\sigma) \frac{1}{(p \cdot q)^2} t_2(\omega, Q^2) \right\}, \quad (3)$$

where  $\omega = 1/x$ .

According to OPE, the current correlation function Eq.(2) is written as follows in the Bjorken limit [5],

$$\begin{aligned} T_{\mu\nu}^A = & -\varepsilon_{\mu\nu\lambda\sigma} q^\lambda \sum_{n:\text{odd}} \left( \frac{2}{Q^2} \right)^n q_{\mu_1} \cdots q_{\mu_{n-1}} \\ & \times \left\{ E_q^n \langle p, s | R_q^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle + \sum_j E_j^n \langle p, s | R_j^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle \right\}. \end{aligned} \quad (4)$$

$R_i$ 's are the composite operators and  $E_i$ 's the corresponding coefficient functions. In Eq.(4),  $R_q$  are the twist 2 operator and others the twist 3 ones. For simplicity, let us consider the flavor non-singlet case.  $R_q$  are explicitly given by the following traceless operators. (Subtractions of trace terms are always understood in the following.)

$$R_q^{\sigma\mu_1 \cdots \mu_{n-1}} = i^{n-1} \bar{\psi} \gamma_5 \gamma^{\{\sigma} D^{\mu_1} \cdots D^{\mu_{n-1}} \} \psi,$$

where  $\{ \dots \}$  denotes the symmetrization over the Lorentz indices between them and  $D^\mu$  is the covariant derivative. (The flavor matrices  $\lambda_i$  for the quark field  $\psi$  are suppressed in this paper.) We have for the twist 3 operators,

$$R_F^{\sigma\mu_1\cdots\mu_{n-1}} = \frac{i^{n-1}}{n} \left[ (n-1) \bar{\psi} \gamma_5 \gamma^\sigma D^{\{\mu_1} \dots D^{\mu_{n-1}\}} \psi - \sum_{l=1}^{n-1} \bar{\psi} \gamma_5 \gamma^{\mu_l} D^{\{\sigma} D^{\mu_1} \dots D^{\mu_{l-1}} D^{\mu_{l+1}} \dots D^{\mu_{n-1}\}} \psi \right] \quad (5)$$

$$R_m^{\sigma\mu_1\cdots\mu_{n-1}} = i^{n-2} m \bar{\psi} \gamma_5 \gamma^\sigma D^{\{\mu_1} \dots D^{\mu_{n-2}} \gamma^{\mu_{n-1}\}} \psi \quad (6)$$

$$R_k^{\sigma\mu_1\cdots\mu_{n-1}} = \frac{1}{2n} (V_k - V_{n-1-k} + U_k + U_{n-1-k}) \quad (7)$$

$m$  in Eq.(6) is the quark mass (matrix).  $V$  and  $U$  in Eq.(7) depend on the gluon field strength  $G_{\mu\nu}$  and the dual tensor  $\tilde{G}^{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}$ , respectively, and given by

$$\begin{aligned} V_k &= i^n g S \bar{\psi} \gamma_5 D^{\mu_1} \dots G^{\sigma\mu_k} \dots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \\ U_k &= i^{n-3} g S \bar{\psi} D^{\mu_1} \dots \tilde{G}^{\sigma\mu_k} \dots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \end{aligned}$$

where  $S$  means the symmetrization over  $\mu_i$  and  $g$  is the strong coupling constant. The operators Eqs.(5 - 7) are not all independent of each other but related through the operators which are proportional to the *equation of motion*,

$$\begin{aligned} R_{eq}^{\sigma\mu_1\cdots\mu_{n-1}} &= i^{n-2} \frac{n-1}{2n} S [\bar{\psi} \gamma_5 \gamma^\sigma D^{\mu_1} \dots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} (i \not{D} - m) \psi \\ &\quad + \bar{\psi} (i \not{D} - m) \gamma_5 \gamma^\sigma D^{\mu_1} \dots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi]. \end{aligned}$$

The following relation is easily obtained :

$$R_F^{\sigma\mu_1\cdots\mu_{n-1}} = \frac{n-1}{n} R_m^{\sigma\mu_1\cdots\mu_{n-1}} + \sum_{k=1}^{n-2} (n-1-k) R_k^{\sigma\mu_1\cdots\mu_{n-1}} + R_{eq}^{\sigma\mu_1\cdots\mu_{n-1}}. \quad (8)$$

Now define the matrix elements of these operators between nucleon states with momentum  $p$  and spin  $s$  by

$$\langle p, s | R_q^{\sigma\mu_1\cdots\mu_{n-1}} | p, s \rangle = -a_n s^{\{\sigma} p^{\mu_1} \dots p^{\mu_{n-1}\}} \quad (9)$$

$$\langle p, s | R_F^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle = -\frac{n-1}{n} d_n (s^\sigma p^{\mu_1} - s^{\mu_1} p^\sigma) p^{\mu_2} \cdots p^{\mu_{n-1}} \quad (10)$$

$$\langle p, s | R_m^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle = -e_n (s^\sigma p^{\mu_1} - s^{\mu_1} p^\sigma) p^{\mu_2} \cdots p^{\mu_{n-1}} \quad (11)$$

$$\langle p, s | R_k^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle = -f_n^k (s^\sigma p^{\mu_1} - s^{\mu_1} p^\sigma) p^{\mu_2} \cdots p^{\mu_{n-1}} \quad (12)$$

$$\langle p, s | R_{eq}^{\sigma\mu_1 \cdots \mu_{n-1}} | p, s \rangle = 0. \quad (13)$$

We have normalized operators such that for a free quark target  $a_n = d_n = e_n = 1$ . On the other hand,  $f_n^k = \mathcal{O}(g^2)$ . The moment sum rule for  $g_1$  and  $g_2$  become using Eq.(9 - 13),

$$\int_0^1 dx x^{n-1} g_1(x, Q^2) = \frac{1}{2} a_n E_q^n(Q^2). \quad (14)$$

$$\begin{aligned} \int_0^1 dx x^{n-1} g_2(x, Q^2) &= -\frac{n-1}{2n} a_n E_q^n(Q^2) \\ &+ \frac{1}{2} \left[ \frac{n-1}{n} d_n E_F^n(Q^2) + e_n E_m^n(Q^2) + \sum_k f_n^k E_k^n(Q^2) \right]. \end{aligned} \quad (15)$$

It is to be noted that from Eq.(8) we have the following constraint,

$$\frac{n-1}{n} d_n = \frac{n-1}{n} e_n + \sum_{k=1}^{n-2} (n-1-k) f_n^k.$$

As mentioned before, we see from Eqs.(5-7) that twist 3 operators can *not* be defined for  $n = 1$ . Namely,  $d_n$ ,  $e_n$ ,  $f_n^k$ 's are identically zero for  $n = 1$  in (15). Also we have a “kinematical” factor  $n - 1$  in front of the contribution from the twist 2 operators in the moment sum rule for  $g_2$  Eq.(15). Therefore the OPE analysis implies that the BC sum rule does not suffer from the radiative corrections at all.

Here note that when expanding  $t_1(\omega, Q^2)$  and  $t_2(\omega, Q^2)$  in Eq.(3) in powers of  $\omega$ , their  $n$ -th coefficients are just equal to the moment sum rules Eq.(14) and Eq.(15) respectively. Therefore if we know  $t_2(\omega, Q^2)$ , we can check whether the above *naïve* argument for  $n = 1$  holds or not. (This is the standard procedure to obtain the higher order corrections for the coefficient functions.) The order of  $\alpha_s$  diagrams contributing to  $T_{\mu\nu}^A$  are shown in Fig.1. The calculation is performed with massive quarks. The

ultraviolet divergences are regularized with momentum cut off  $\Lambda$ . To regularize the infrared singularities, we give a mass  $\lambda$  to the gluon. Note that the collinear singularities are already regulated by quark mass. We choose the on-shell renormalization scheme. Although we have calculated the diagrams without making any approximations for  $m^2/Q^2$ , we present only the results after taking the limit  $m^2/Q^2 \rightarrow 0$  in accordance with the fact that we kept only the leading twist terms in the OPE. The contributions from each diagrams contain current non-conserved terms which cancel out after adding all contributions. So we will neglect such terms in the following results.

The results for each diagrams read ( $C_F (= 4/3)$  is the Casimir operator for quarks):

(a) Born + self-energy contribution of Fig.1a :

$$\begin{aligned} t_1^{(a)} &= \sum_n \omega^n \left[ 1 + \frac{g^2}{16\pi^2} C_F \left\{ \ln \frac{Q^2}{m^2} + 3 + \frac{1}{n} - \sum_{r=1}^n \frac{1}{r} + 2 \ln \frac{\lambda^2}{m^2} \right\} \right], \\ t_2^{(a)} &= 0. \end{aligned}$$

(b) vertex contribution of Fig.1b :

$$\begin{aligned} t_1^{(b)} &= \frac{g^2}{8\pi^2} C_F \sum_n \omega^n \left[ \left( -1 - 2 \sum_{r=2}^n \frac{1}{r} \right) \ln \frac{Q^2}{m^2} - 4 + \frac{1}{n} + 2 \sum_{r=1}^n \frac{1}{r} \right. \\ &\quad \left. - 2 \sum_{r=1}^n \frac{1}{r^2} - 2 \sum_{s=1}^n \frac{1}{s} \sum_{r=1}^s \frac{1}{r} - 2 \ln \frac{\lambda^2}{m^2} \right], \\ t_2^{(b)} &= \frac{g^2}{8\pi^2} C_F \sum_n \omega^n \left[ \left( -1 + \frac{1}{n} \right) \ln \frac{Q^2}{m^2} + \frac{1}{2} - \frac{3}{n} + \frac{1}{n^2} + \left( \frac{1}{2} + \frac{1}{n} \right) \sum_{r=1}^n \frac{1}{r} \right]. \end{aligned}$$

(c) box contribution of Fig.1c :

$$\begin{aligned} t_1^{(c)} &= \frac{g^2}{8\pi^2} C_F \sum_n \omega^n \left[ \frac{1}{n(n+1)} \ln \frac{Q^2}{m^2} - \frac{4}{n} + \frac{4}{n+1} + \frac{1}{n^2} \right. \\ &\quad \left. - \frac{2}{(n+1)^2} + \left( 2 + \frac{1}{n(n+1)} \right) \sum_{r=1}^n \frac{1}{r} + \ln \frac{\lambda^2}{m^2} \right], \\ t_2^{(c)} &= \frac{g^2}{8\pi^2} C_F \sum_n \omega^n \left[ \left( \frac{2}{n+1} - \frac{1}{n} \right) \ln \frac{Q^2}{m^2} + \frac{3}{n} - \frac{6}{n+1} + \frac{1}{n^2} \right. \\ &\quad \left. + \frac{4}{(n+1)^2} + \left( \frac{2}{n+1} - \frac{1}{n} \right) \sum_{r=1}^n \frac{1}{r} \right]. \end{aligned}$$

Now the entire expressions for  $t_1$  and  $t_2$  in which the above three contributions are added together are:

$$\begin{aligned}
t_1 &= \sum_n \omega^n \left[ 1 + \frac{g^2}{8\pi^2} C_F \left\{ -\frac{1}{2} \left( 1 - \frac{2}{n(n+1)} + 4 \sum_{r=2}^n \frac{1}{r} \right) \ln \frac{Q^2}{m^2} \right. \right. \\
&\quad \left. -\frac{5}{2} - \frac{5}{2n} + \frac{4}{n+1} + \frac{1}{n^2} - \frac{2}{(n+1)^2} \right. \\
&\quad \left. + \left( \frac{7}{2} + \frac{1}{n(n+1)} \right) \sum_{r=1}^n \frac{1}{r} - 2 \sum_{r=1}^n \frac{1}{r^2} - 2 \sum_{r=s}^n \frac{1}{s} \sum_{r=1}^s \frac{1}{r} \right\} \right], \\
t_2 &= \frac{g^2}{8\pi^2} C_F \sum_n \omega^n \left[ -\frac{1}{2} \frac{2(n-1)}{n+1} \ln \frac{Q^2}{m^2} \right. \\
&\quad \left. + \frac{1}{2} - \frac{6}{n+1} + \frac{4}{(n+1)^2} + \frac{n+5}{2(n+1)} \sum_{r=1}^n \frac{1}{r} \right]. \quad (16)
\end{aligned}$$

Eq.(16) and its interpretation is in agreement with the result of Ref. [9]. However The result Eq.(16) is in agreement with that of Ref. [10]. The first moment ( $n = 1$ ) which corresponds to the BC sum rule vanishes. So our calculation reconfirms the validity of the BC sum rule and shows that the OPE analysis is consistent with the QCD perturbation theory.

In conclusion we have calculated the virtual Compton scattering amplitude at order  $\alpha_s$  and shown that the BC sum rule does not receive any corrections in perturbative QCD based on the OPE. In this respect, here we note that the BC sum rule is not only protected from QCD radiative corrections but also free from target mass effects [11].

Finally, we expect that future experiments on  $g_2$  will confirm the BC sum rule in its original form.

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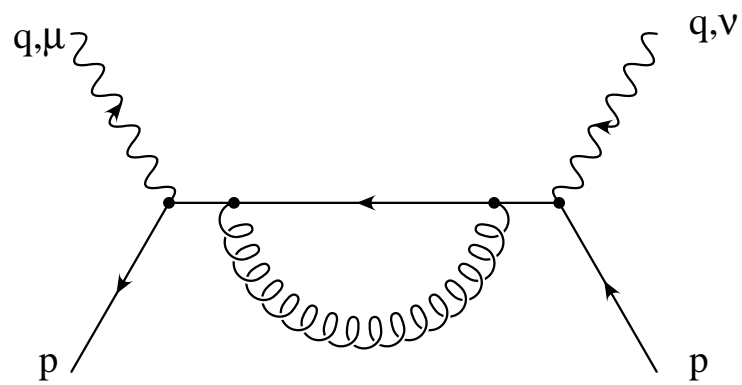
## Figure Caption

Fig.1

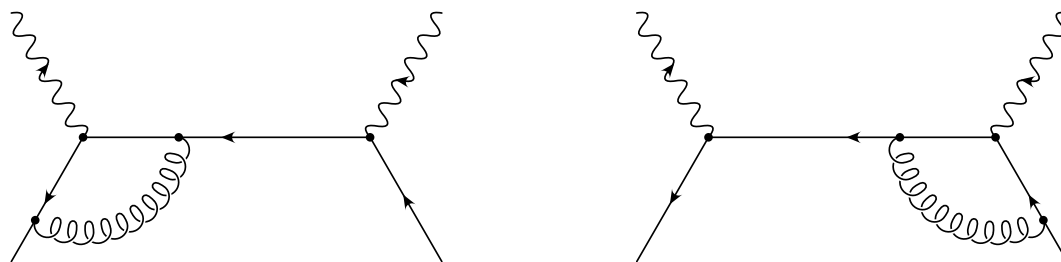
Diagrams (a) self-energy, (b) vertex and (c) box, contributing to the current correlation function at order  $g^2$ .

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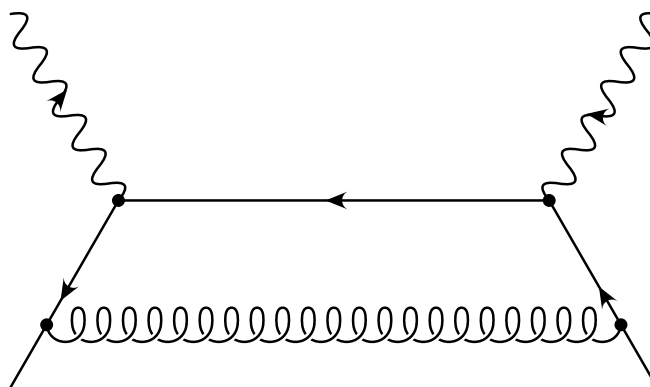
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(a)



(b)



(c)

**Fig.1**